The geometry of finite Blaschke products: some duality results

Masayo Fujimura
Department of Mathematics, National Defense Academy

Abstract

We study geometrical properties of finite Blaschke products. For a Blaschke product B of degree d, the interior curve and the exterior curve are defined. Daepp *et al.* proved that the interior curve of B of degree 3 forms an ellipse. However, in the case of $d > 3$, the properties of the interior curve including of the degree are still unknown. On the other hand, it is known that the exterior curve is an algebraic curve of degree at most $d-1$. In this paper, we give a relation between the interior curve and the exterior curve.

1 Blaschke Products

A *Blaschke product* of degree d is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k} z} \qquad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical.*

For a Blaschke product B of degree d, set $f_1(z) = e^{-\frac{\theta}{d} i} z$, and $f_2(z) = \frac{z-(-1)^d a_1 \cdots a_d e^{i\theta}}{1-(-1)^d \overline{a_1 \cdots a_d e^{i\theta}} z}$. Then, the composition $f_2 \circ B \circ f_1$ is a canonical one, and geometrical properties with respect to preimages of these two Blaschke products B and $f_2 \circ B \circ f_1$ are same. Hence, we only need to consider a canonical Blaschke product for the following discussions. Here, we remark that there are d distinct preimages $z_1, \cdots, z_d$ of $\lambda \in \partial\mathbb{D}$ by B because the derivative of B has no zeros on $\partial\mathbb{D}$ (for instance, see [Mas13]).

In this paper, we will discuss geometrical properties of curves defined by the preimages of points on the unit circle under B.

2 The interior curves and exterior curves

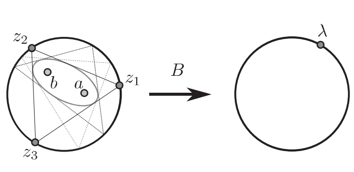
For Blaschke product B of degree d and $\lambda \in \partial\mathbb{D}$, let ℓ_λ be the set of lines joining each distinct two preimages in $B^{-1}(\lambda)$. Then, the envelope of the family of lines $\{\ell_\lambda\}_\lambda$ called the *interior curve* associated with B.

The following is a beautiful property of the interior curve associated with Blaschke product B of degree 3.

Theorem 1 (U. Daepp, P. Gorkin, and R. Mortini [DGM02])

Let B be a canonical Blaschke product of degree 3 with zeros $0, a,$ and b. For $\lambda \in \partial\mathbb{D}$, let $z_1, z_2,$ and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation

$$|z - a| + |z - b| = |1 - \overline{a}b|.$$



The ellipse E corresponds to "the inner ellipse" in Poncelet's Theorem.

For a canonical Blaschke product B of degree d and a point λ on the unit circle, let L_λ be the set of d lines tangent to $\partial\mathbb{D}$ at the d preimages of λ. Then, the trace of the intersection points of each two elements in L_λ as λ ranges over the unit circle, called the *exterior curve* associated with B.

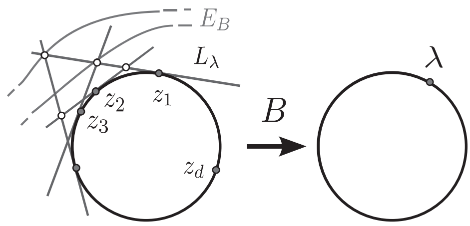


Figure 1: The exterior curve is defined by the trace of the intersection points of tangent lines.

When the degree is low, the interior curve and the exterior curve are described concretely, as follows.

1. For a canonical Blaschke product $B(z) = z\dfrac{z-a}{1-\overline{a}z}$ of degree 2 with zeros 0 and $a(\neq 0)$,
   - The interior curve is the point a (see [DGM02]).
   - The exterior curve is the line $\overline{a}z + a\overline{z} - 2 = 0$ (see [Fuj17]).
2. For a canonical Blaschke product $B(z) = z\dfrac{z-a}{1-\overline{a}z}\dfrac{z-b}{1-\overline{b}z}$ of degree 3,
   - The interior curve is the ellipse $|z-a| + |z-b| = |1-\overline{a}b|$ (see [DGM02]).
   - The exterior curve is the conic

$$\overline{a}\overline{b}z^2 + (-|ab|^2 + |a+b|^2 - 1)z\overline{z} + ab\overline{z}^2 - 2(\overline{a}+\overline{b})z - 2(a+b)\overline{z} + 4 = 0. \tag{1}$$

   Moreover, the above curve is a non-degenerate conic, i.e. either an ellipse, a circle, a parabola, or a hyperbola (see [Fuj17]).

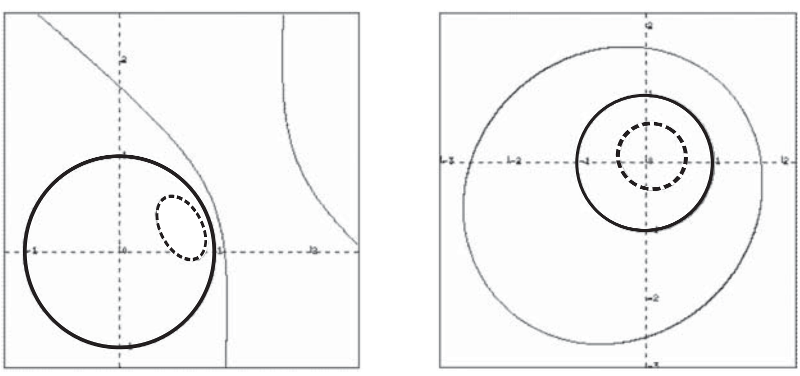


Figure 2: In the case of degree 3: $(a,b) = (\frac{1}{2} + \frac{1}{2}i, \frac{4}{5})$ (left), $(a,b) = (\frac{1}{5}i, \frac{1}{5})$ (right).
The thick circle is the unit circle, the dotted curves and the thin curves indicate the interior and the exterior curves, respectively.

3. For a canonical Blaschke product $B(z) = z\dfrac{z-a}{1-\overline{a}z}\dfrac{z-b}{1-\overline{b}z}\dfrac{z-c}{1-\overline{c}z}$ of degree 4,

- The interior curve is defined by the equation of degree 6. To compute a defining equation of this curve we use Risa/Asir, a symbolic computation system. The output of the result is about 200Kb as a text file (see [Fuj13]). So we cannot describe it here.
- The exterior curve is written as follows (see [Fuj17])

$$\begin{aligned}&\overline{\sigma_3}z^3 + (\sigma_1\overline{\sigma_2} - \sigma_2\overline{\sigma_3} - \overline{\sigma_1})z^2\overline{z} - (\sigma_1 - \sigma_2\overline{\sigma_1} + \sigma_3\overline{\sigma_2})z\overline{z}^2 + \sigma_3\overline{z}^3\\ &- 2\overline{\sigma_2}z^2 - (2\sigma_1\overline{\sigma_1} - 2\sigma_3\overline{\sigma_3} - 4)z\overline{z} - 2\sigma_2\overline{z}^2 + 4\overline{\sigma_1}z + 4\sigma_1\overline{z} - 8 = 0,\end{aligned}$$

  where σ_k are the elementary symmetric polynomials on three variables $a,\ b,\ c$ of degree k ($k = 1, 2, 3$), i.e.

$$\sigma_1 = a + b + c,\ \ \sigma_2 = ab + bc + ca \quad \text{and} \quad \sigma_3 = abc.$$

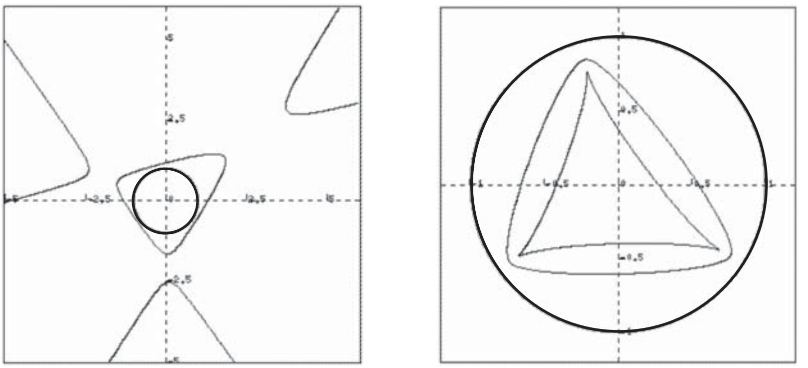


Figure 3: In the case of degree 4: $(a, b, c) = (-0.72 - 0.51i, 0.75 - 0.47i, -0.22 + 0.83i)$
The left figure indicates the exterior curve, and the right figure indicates the interior curve. The thick circles in both figures are the unit circles.

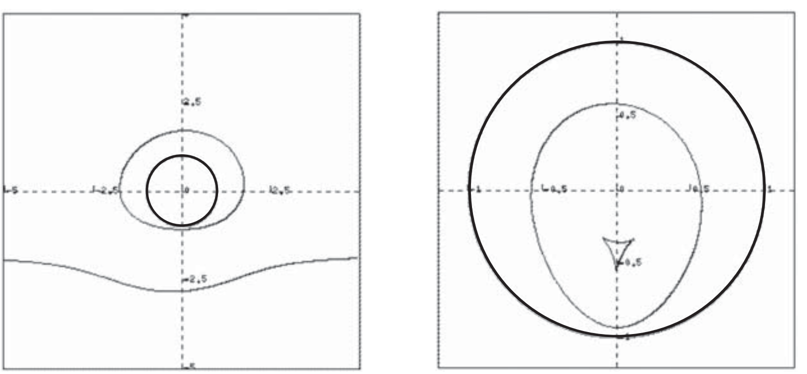


Figure 4: In the case of degree 4: $(a, b, c) = (-0.8i, 0.1 - 0.1i, -0.1 + 0.2i)$.
The left figure indicates the exterior curve, and the right figure indicates the interior curve. The thick circles in both figures are the unit circles.

4. For a canonical Blaschke product of degree 5 with zeros $0, a, b, c$ and d,
   - The defining equation of the interior curve is difficult to compute, even though we use symbolic computation systems.
   - The exterior curve is written as follows

$$
\begin{aligned}
&\overline{\sigma_4}z^4 + (\sigma_1\overline{\sigma_3} - \overline{\sigma_2} - \sigma_2\overline{\sigma_4})z^3\overline{z} - (\sigma_1\overline{\sigma_1} - \sigma_2\overline{\sigma_2} + \sigma_3\overline{\sigma_3} - \sigma_4\overline{\sigma_4} - 1)z^2\overline{z}^2 \\
&+ (\sigma_3\overline{\sigma_1} - \sigma_4\overline{\sigma_2} - \sigma_2)z\overline{z}^3 + \sigma_4\overline{z}^4 - 2\overline{\sigma_3}z^3 + 2(2\overline{\sigma_1} - \sigma_1\overline{\sigma_2} + \sigma_3\overline{\sigma_4})z^2\overline{z} \\
&- 2(\sigma_2\overline{\sigma_1} - 2\sigma_1 - \sigma_4\overline{\sigma_3})z\overline{z}^2 - 2\sigma_3\overline{z}^3 + 4\overline{\sigma_2}z^2 + 4(\sigma_1\overline{\sigma_1} - \sigma_4\overline{\sigma_4} - 3)z\overline{z} \\
&+ 4\sigma_2\overline{z}^2 - 8\overline{\sigma_1}z - 8\sigma_1\overline{z} + 16 = 0, \qquad (2)
\end{aligned}
$$

   where σ_k are the elementary symmetric polynomials on four variables $a,\ b,\ c,\ d$ of degree k ($k = 1, \cdots, 4$) (see [Fuj18]).

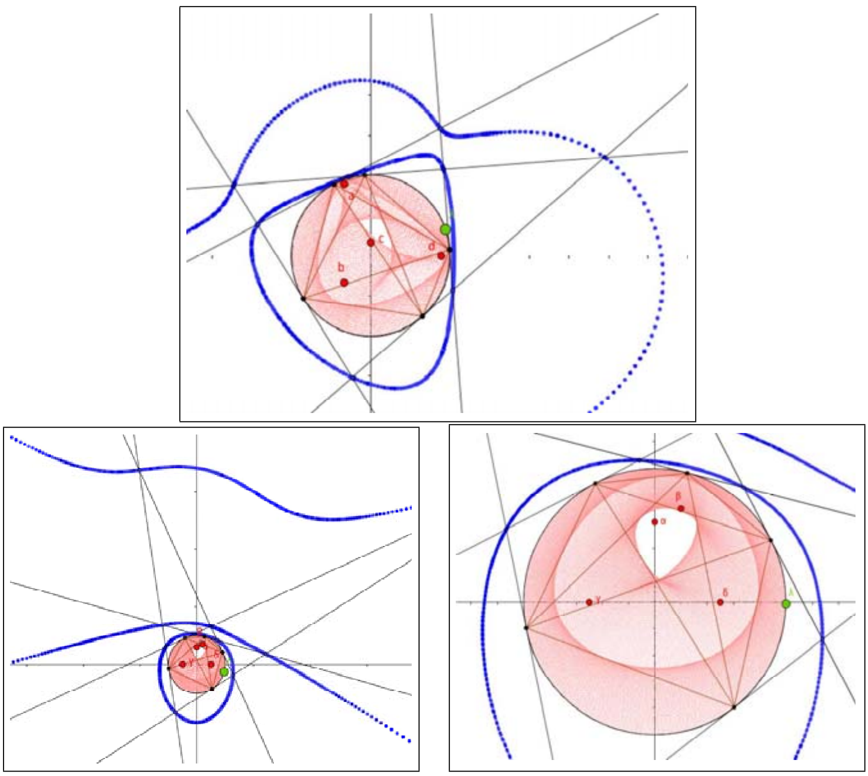


Figure 5: In the case of degree 5: $(a, b, c, d) = (-\frac{1}{3} + \frac{8}{9}i, -\frac{1}{3} - \frac{1}{3}i, \frac{1}{6}i, \frac{8}{9})$ (upper), $(0.6i, 0.2+0.7i, -0.5, 0.5)$ (lower), The envelope on the lower right figure is the interior curve corresponding to the lower left. We will give the defining equation of the envelope in the upper figure in Example 2 (cf. Figure 7).

For the degree of the interior curve, we obtained the following.

Theorem 2 ([Fuj17])
Let B be a canonical Blaschke product of degree d. Then, the exterior curve E_B is an algebraic curve of degree at most $d - 1$.

3 Some duality

Is there any relevance between the interior curve and the exterior curve? The following theorem gives a solution to this question.

Theorem 3 ([Fuj18])
Let B be a canonical Blaschke product of degree d, and E_B^* the dual curve of the homogenized exterior curve E_B. Then, the interior curve is given by

$$I_B: \ u_B^*(-z) = 0,$$

where $u_B^*(z) = 0$ is a defining equation of the affine part of E_B^*.

Equivalently, the converse also holds.

Corollary 4
Let B be a canonical Blaschke product of degree d, and I_B^* be the dual curve of the homogenized interior curve I_B. Then, the exterior curve is given by

$$E_B: \ v_B^*(-z) = 0,$$

where $v_B^*(z) = 0$ is a defining equation of the affine part of I_B^*.

Remark 1
For every two preimages w_1, w_2 of $\lambda \in \partial\mathbb{D}$ under B, we can consider the intersection point p of two lines tangent to the unit circle at these two points. Then, the line joining w_1 and w_2 is the polar of the unit circle with respect to the pole p.

Example 1
Consider the Blaschke product

$$B(z) = z \cdot \frac{z - \frac{2}{3}}{1 - \frac{2}{3}z} \cdot \frac{z - (\frac{1}{3} + \frac{2}{3}i)}{1 - (\frac{1}{3} - \frac{2}{3}i)z} \tag{3}$$

of degree 3.

Substituting the zero points $a = \frac{2}{3}$, $b = \frac{1}{3} + \frac{2}{3}i$ to the equation (1), we have a defining equation of the exterior curve

$$(18 - 36i)z^2 + 16z\overline{z} - (162 - 108i)z + (18 + 36i)\overline{z}^2 - (162 + 108i)\overline{z} + 324 = 0.$$

Then, the homogenization of the exterior curve is written as

$$u_B(z:t) = (18 - 36i)z^2 + 16z\overline{z} - (162 - 108i)tz + (18 + 36i)\overline{z}^2 - (162 + 108i)t\overline{z} + 324t^2 = 0.$$

Here, we consider the following system of equations,

$$\overline{z} = 2\frac{\partial}{\partial \zeta}u_B(\zeta : \tau), \quad z = 2\frac{\partial}{\partial \overline{\zeta}}u_B(\zeta : \tau), \quad t = \frac{\partial}{\partial \tau}u_B(\zeta : \tau), \quad u_B(\zeta : \tau) = 0. \tag{4}$$

Eliminating ζ, τ from (4) we have the defining equation of the dual curve

$$\begin{aligned} u_B^*(z:t) =& (-2187 + 2916i)z^2 - 13770z\overline{z} - (11016 - 6048i)tz - (2187 + 2916i)\overline{z}^2 \\ & - (11016 + 6048i)t\overline{z} - 6224t^2 = 0. \end{aligned}$$

Taking the reflection of the affine part of the above through the origin, we have

$$(-2187 + 2916i)z^2 - 13770z\overline{z} + (11016 - 6048i)z - (2187 + 2916i)\overline{z}^2 + (11016 + 6048i)\overline{z} - 6224 = 0. \tag{5}$$

This equation gives a defining equation of the interior curve from Theorem 3.

On the other hand, from Theorem 1, the interior curve I_B is the ellipse

$$\left|z - \frac{2}{3}\right| + \left|z - \left(\frac{1}{3} + \frac{2}{3}i\right)\right| = \left|1 - \frac{2}{3}\left(\frac{1}{3} + \frac{2}{3}i\right)\right|.$$

We can check that the above equation can be expressed as (5).

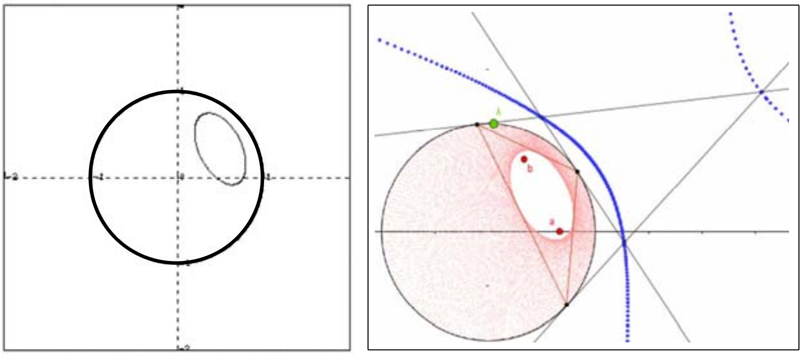


Figure 6: The interior curve I_B of Blaschke product (3) of degree 3: The thick circle is the unit circle. The interior curve I_B in the left figure coincides with the envelope in the right figure.

Example 2
Consider the Blaschke product

$$B(z) = z \cdot \frac{z-(\frac{1}{3}+\frac{8}{9}i)}{1-(\frac{1}{3}-\frac{8}{9}i)z} \cdot \frac{z-(-\frac{1}{3}-\frac{1}{3}i)}{1-(-\frac{1}{3}+\frac{1}{3}i)z} \cdot \frac{z-\frac{1}{6}i}{1+\frac{1}{6}iz} \cdot \frac{z-\frac{8}{9}}{1-\frac{8}{9}z} \tag{6}$$

of degree 5.

It is difficult to calculate the defining equation of the interior curve directly, but we can obtain the equation by using the result of Theorem 3 as follows.

Substituting the zero points $a=-\frac{1}{3}+\frac{8}{9}i,\ b=-\frac{1}{3}-\frac{1}{3}i,\ c=\frac{1}{6}i,\ d=\frac{8}{9}$ *to the equation* (2), *we have a defining equation of the exterior curve*

$$\begin{aligned}
&(58320-128304i)z^4+((409005+1248381i)\overline{z}+(-1320948-831060i))z^3\\
&\quad+(1052930\overline{z}^2+(1076664-5064102i)\overline{z}+(-2361960-2939328i))z^2+((409005-1248381i)\overline{z}^3\\
&\quad+(1076664+5064102i)\overline{z}^2-20691404\overline{z}+(-3779136+12282192i))z+(58320+128304i)\overline{z}^4\\
&\quad-(1320948-831060i)\overline{z}^3-(2361960-2939328i)\overline{z}^2-(3779136+12282192i)\overline{z}+34012224=0.
\end{aligned} \tag{7}$$

In this case, it is hard to calculate the dual curve using the same procedure as Example 1. Using the condition of a line in the projective plane tangent to the homogenization of (7), *we can calculate the defining equation of the dual curve* $u_B^*(z,t)=0$. *Taking the reflection of the dual curve through the origin, we have the following defining equation of the interior curve*

$$\begin{aligned}
&I_B:\ u_B^*(-z,1)=\\
&-(1539979487119056105333138301285317120 0+931605943773071695773090970567888416 00i)z^{12}\\
&+((-627338421372704312628743046091308312768-1079251262618833882035510395164692762 24i)\overline{z}+\\
&(-421499772859600852951670495185234247424+149551173500954916714838575401655831168i))z^{11}\\
&+((182365076985788368426087874959978585406 4-204232122825646652127015402451745092924 8i)\overline{z}^2+\\
&(914727195587267285125696970332823606304+226300143339243452479186332200138175907 2i)\overline{z}+\\
&(666584668669945833556382418834060324864+787884861656745276853950006314125400352i))z^{10}\\
&+((244725126147788719979626610846120719344+255907835511282452553098613394847321606 4i)\overline{z}^3+\\
&(-205187342708248363257515788365592924867 2-316355618630789031576744064579196353996 8i)\overline{z}^2+\\
&(327418284978940544637429834400733918236 8-342528902604886373448256420170771044995 2i)\overline{z}+\\
&(409779582109122705700495209797036379456-160691300227971796393265431079470138732 8i))z^9\\
&+((269714783698348859595549200818004309604-223989021736992128635536389250779543836 8i)\overline{z}^4+\\
&(431384656390404126683340488133847643280+242789373776976314466904735949407921816 08i)\overline{z}^3+\\
&(-951542642082571035660652024214648462613 6-112790845948398778965224907826143951196 8i)\overline{z}^2+\\
&(-878686298296293277921395842565197793508 8-218942287072339777501150741655791270008 0i)\overline{z}+\\
&(-237513127364772451445583829967802643238 4+112496020479043777726688237486177239725 6i))z^8
\end{aligned}$$

$$
\begin{aligned}
&+((-1947598686318801570273994841617761245232-7531942676699568588046820819137788848064i)\overline{z}^5+\\
&(-3023978766749679508275272939084194393536 0-1902977262163323268623686670415554160573 2i)\overline{z}^4+\\
&(5081248434884728862062262852546198734270 8-1689814336065331921938843221671171653052 8i)\overline{z}^3+\\
&(5005101875654457945642036207038450784984-1381286175504319498890581778800105374680i)\overline{z}^2+\\
&(5348711907367394432639391022669936055604+1246199612913923359187983163619111433066 8i)\overline{z}+\\
&(3491483141002171428210091554794478901680+1904777487610891917999689336184998412840i))z^7\\
&+(1895004831582970999208307455151492075477 6\overline{z}^6+(-7013865454572586840637247840786893322768+\\
&2273509492287022320502172071115702093922 0i)\overline{z}^5+(-3507907877814963637860883830853699680188 1+\\
&1408291300533412498686334069919430929407 2i)\overline{z}^4+(3341667917987193061309849064283521257812-\\
&3397939330969119120025626425833228568741 4i)\overline{z}^3+(-1926981120353425776944487903092525362656 9+\\
&3957698559263561501836858704341055858582i)\overline{z}^2+(6521838892366202671885062016228944485850-\\
&9677658405405658271359826691169142923530i)\overline{z}+(-5994535193071307397134984219430880880 64-\\
&4741925438645558078073192742296840735834i))z^6\\
&+((-1947598686318801570273994841617761245232+7531942676699568588046820819137788848064i)\overline{z}^7+\\
&(-7013865454572586840637247840786893322768-2273509492287022320502172071115702093922 0i)\overline{z}^6+\\
&850626339519843628746736599973325527356 38\overline{z}^5+(-8836668873476677766074139484988861021848+\\
&7069055504978597445185330292552502258768 2i)\overline{z}^4+(-1500658387823002379694635850763404874134 6+\\
&1210385393381534472431772229497446703230 4i)\overline{z}^3+(-1976282917815502366084498231789696358 64+\\
&1767010399017463232691797935345171990030 2i)\overline{z}^2+(-6947816879763396712491978002840319033864-\\
&1252596000340792840677044068629221642892i)\overline{z}+(-3161192041331701773380546212189414967220+\\
&2904376175710197377287691186476134380424i))z^5\\
&+((2697147836983488595955492008180043096 04+2239890217369921286355363892507795438368i)\overline{z}^8+\\
&(-3023978766749679508275272939084194393536 0+1902977262163323268623686670415554160573 2i)\overline{z}^7+\\
&(-3507907877814963637860883830853699680188 1-1408291300533412498686334069919430929407 2i)\overline{z}^6+\\
&(-8836668873476677766074139484988861021848-7069055504978597445185330292552502258768 2i)\overline{z}^5-\\
&2461967219502671157887544364504680154331 0\overline{z}^4+(2583942080607875133988385335625023063001 4-\\
&4818220049999160052448492830781586240799 2i)\overline{z}^3+(1656581424043653495520598132357555573257 5-\\
&7366232959966176285449733595895897753850i)\overline{z}^2+(3120867994772439269914301236374195834918+\\
&3001155051479635543405883654345014427130i)\overline{z}+(3066925768691058290442360773341067114988+\\
&6427948741185256364435468032744628004 54i))z^4\\
&+((2447251261477887199796266108461207193 44-2559078355112824525530986133948473216064i)\overline{z}^9+\\
&(4313846563904041266833404881338476432 80-2427893737769763144669047359494079218160 8i)\overline{z}^8+\\
&(5081248434884728862062262852546198734270 8+1689814336065331921938843221671171653052 8i)\overline{z}^7+\\
&(3341667917987193061309849064283521257812+3397939330969119120025626425833228568741 4i)\overline{z}^6+\\
&(-1500658387823002379694635850763404874134 6-1210385393381534472431772229497446703230 4i)\overline{z}^5+\\
&(2583942080607875133988385335625023063001 4+4818220049999160052448492830781586240799 2i)\overline{z}^4-\\
&1464820206410569953321224664693669189541 8\overline{z}^3+(-1323177577047716183931070891204526498758 2+\\
&9356713378348101232683143990522557242018i)\overline{z}^2+(-2000428128405417278320958680496643523176-\\
&1444605972940058648611879649198222629164i)\overline{z}+(-1007836384993440979534490537203566474900-\\
&1705534009383936842007186243637606082148i))z^3\\
&+((1823650769857883684260878749559978585406 4+2042321228256466521270154024517450929248i)\overline{z}^{10}+\\
&(-2051873427082483632575157883655929248672+3163556186307890315767440645791963539968i)\overline{z}^9+\\
&(-9515426420825710356606520242146484626136+1127908459483987789652249078261439511968i)\overline{z}^8+\\
&(5005101875654457945642036207038450784984+1381286175504319498890581778800105374680i)\overline{z}^7+\\
&(-1926981120353425776944487903092525362656 9-3957698559263561501836858704341055858582i)\overline{z}^6+\\
&(-1976282917815502366084498231789696358 64-1767010399017463232691797935345171990030 2i)\overline{z}^5+\\
&(1656581424043653495520598132357555573257 5+7366232959966176285449733595895897753850i)\overline{z}^4+\\
&(-1323177577047716183931070891204526498758 2-9356713378348101232683143990522557242018i)\overline{z}^3+\\
&7587791904014055406215789564437793454788\overline{z}^2+(1669498981482568421290894827287110243020-\\
&5760177003177982747036635885913907639 64i)\overline{z}+(-1313543317674739620281790027343453249 32+\\
&9505966145961388888398836215849992250 32i))z^2
\end{aligned}
$$

$$
\begin{aligned}
&+((-6273384213727043126287430460913083127 68+10792512626188338820355103951646927622 4i)\overline{z}^{11}+\\
&(9147271955872672851256969703328236063 04-22630014333924345247918633220013817590 72i)\overline{z}^{10}+\\
&(32741828497894054463742983440073391823 68+34252890260488637344825642017077104499 52i)\overline{z}^{9}+\\
&(-87868629829629327792139584256519779350 88+21894228707233977750115074165579127000 80i)\overline{z}^{8}+\\
&(53487119073673944326393910226699360556 04-12461996129139233591879831636191114330 668i)\overline{z}^{7}+\\
&(65218388923662026718850620162289444858 50+96776584054056582713598266911691429235 30i)\overline{z}^{6}+\\
&(-69478168797633967124919780028403190338 64+12525960003407928406770440686292216428 92i)\overline{z}^{5}+\\
&(31208679947724392699143012363741958349 18-30011550514796355434058836543450144271 30i)\overline{z}^{4}+\\
&(-20004281284054172783209586804966435231 76+14446059729400586486118796491982226291 64i)\overline{z}^{3}+\\
&(16694989814825684212908948272871102430 20+57601770031779827470366358859139076396 4i)\overline{z}^{2}-\\
&1418342883157293423385575529720968194376\overline{z}+(2071437357636350213593511895113457377 28-\\
&1964628023943234058219270824375417608 16i))z\\
&+(-15399794871190561053331383012853171200+9316059437730716957730909705678884160 0i)\overline{z}^{12}+\\
&(-4214997728596008529516704951852342474 24-14955117350095491671483857540165583116 8i)\overline{z}^{11}+\\
&(6665846686699458335563824188340603248 64-7878848616567452768539500063141254003 52i)\overline{z}^{10}+\\
&(4097795821091227057004952097970363794 56+16069130022797179639326543107947013873 28i)\overline{z}^{9}+\\
&(-23751312736477245144558382996780264323 84-11249602047904377772668823748617723972 56i)\overline{z}^{8}+\\
&(34914831410021714282100915547944789016 80-19047774876108919179996893361849984128 40i)\overline{z}^{7}+\\
&(-5994535193071307397134984219430880880 64+47419254386455580780731927422968407358 34i)\overline{z}^{6}+\\
&(-31611920413317017733805462121894149672 20-29043761757101973772876911864761343804 24i)\overline{z}^{5}+\\
&(30669257686910582904423607733410671149 88-6427948741185256364435468032744628004 54i)\overline{z}^{4}+\\
&(-10078363849934409795344905372035664749 00+17055340093839368420071862436376060821 48i)\overline{z}^{3}+\\
&(-1313543317674739620281790027343453249 32-9505966145961388888398836215849992250 32i)\overline{z}^{2}+\\
&(2071437357636350213593511895113457377 28+1964628023943234058219270824375417608 16i)\overline{z}-\\
&3528718947937346242540257057266127259 2=0.
\end{aligned}
$$

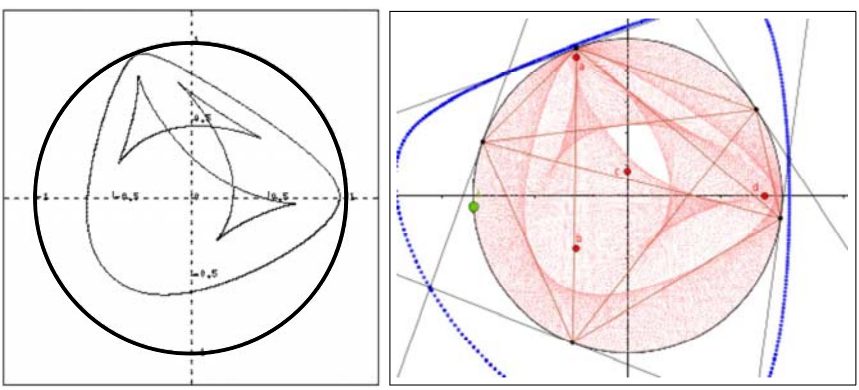


Figure 7: The interior curve of Blaschke product (6) of degree 5: The thick circle is the unit circle. We remark that the interior curve in the left figure coincides with the envelope in the right figure (cf. the upper figure in Figure 5).

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Department of Mathematics
National Defense Academy
Yokosuka 239-8686
JAPAN
E-mail address: `masayo@nda.ac.jp`

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